JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 27, No. 1, January–February 2004

Nonlinear Recursive Minimum Model Error Estimation

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A novel approach is presented for nonlinear recursive estimation of dynamic discrete-time systems with model uncertainties. The formulation of this estimation problem is the same as for minimum model error estimation, but the solution of the nonlinear two-point boundary-value problem is obtained using the technique of invariant imbedding so that the resulting algorithm can be implemented in real time. The advantage of the proposed estimation over other invariant-imbedding-based sequential estimators is that the covariance constraint condition developed in the minimum model error estimation is used to choose the weighting matrix and model errors are modeled as time-varying parameters whose variations are chosen to be minimal. The simulation results show that the proposed method is effective and feasible in obtaining optimal estimation for nonlinear systems with nonlinear measurements in the face of significant model errors and large initial condition errors.

Introduction

NONLINEAR estimation of dynamic systems with model uncertainties has always been a difficult problem as reflected by the many articles in the literature. Among numerous approaches, the first-order extended Kalman filter (EKF) is most popular due to its simplicity. In the EKF, model uncertainty is assumed to be a zero-mean Gaussian process with known variance. Nevertheless, estimates may be biased or divergent in actual practice because of linearization and initial condition error. Recently, some approaches have been developed to address the approximations used in the EKF with nonlinear observations to improve the convergence of the estimates. Among them, the two-step optimal estimator[1] (TSOE) divides the nonlinear estimation problem into a linear estimation using the Kalman filter and an optimization procedure to get the optimal estimates of the system states. Lisano[2] pointed out that the TSOE is only effective and advantageous over the EKF for initial conditions with large errors. Set-valued nonlinear estimation[3] employs the Galerkin approximation to solve Kolmogorov's equation for propagating a continuous nonlinear dynamic system and for implementing a discrete-time measurement update. The unscented Kalman filter (UKF) (see Ref. 4) extends the unscented transformation to the recursive nonlinear estimation of dynamic systems. Like the EKF, the UKF approximates the state distribution by a Gaussian random variable represented by a set of sample points that capture its true mean and covariance. When propagated through the true nonlinear system, the sample points capture the posterior mean and covariance accurately to a third-order Taylor series expansion. As a result, the UKF substantially improves the accuracy of the nonlinear estimation over the EKF estimate with the same level of computational complexity.

For physical systems, model uncertainties may result from uncertain variables such as external disturbances, modeling errors, or parametric variations. They are generally referred to as model errors or process noise. In the mentioned methods, model errors are assumed to be Gaussian distributed random variables with zero mean and known variance. However, it is often hard to model their dynamics as known stochastic processes. Therefore, the aforementioned methods may fail to perform optimally or to converge.

Historically, the problem of nonlinear estimation for dynamic systems with statistically unknown model errors or measurement errors was addressed by minimizing a nonstatistical least-squares criterion, and thus, a nonlinear two-point boundary-value problem (TPBVP) resulted. Approaches to the solution of the resulting TPBVP produced various estimators. Detchmendy and Sridhar,[5] Bellman et al.,[6] Pearson,[7] and Sage[8] developed their own sequential estimators using the method of invariant imbedding. Their simulation studies show that these online estimators were feasible for nonlinear estimation with model uncertainties. However, no methods for choosing the weighting matrices in their criteria were put forward. In 1988, Mook and Junkins[9] proposed a batch estimator called minimum model error (MME) estimation utilizing the method of multiple shooting. In the MME approach, estimates of system states and model errors are obtained as part of the solution of the resulting TPBVP. The covariance constraint concept was introduced as the necessary condition for choosing the weighting matrix. An existence and uniquencess proof for the solution of the MME used the multiple shooting approach.[10] Convergence difficulties near the minimum were overcome using a technique to linearize the state and costate dynamic functions around the nominal state and model error determined by the gradient search.[11] Recently, Crassidis and Markley[12] proposed a predictive filter for nonlinear recursive estimation of systems with model uncertainties based on the duality that exists between the predictive controller[13] for nonlinear systems and a general estimation problem. The criterion for the algorithm is the weighted sum square of the measurement minus estimate residuals plus the weighted sum square of the model correction term. The output estimate is expanded as a Taylor series, and the criterion is minimized with respect to the model error to calculate the model error solution and state estimates. Compared with the MME, however, the predictive filter does not optimize the estimate globally, and its convergence properties are closely related with the sampling interval.

This paper presents a recursive method for nonlinear estimation of dynamic systems with model uncertainties. The formulation of the estimation problem is the same as that of the MME estimator, but the solution to the nonlinear TPBVP is obtained using the technique of discrete invariant imbedding. The new method, nonlinear recursive minimum model error (NRMME) estimation, can be implemented recursively. The advantage of the NRMME estimation over the previous invariant-imbedding-based sequential estimators is that the covariance constraint condition developed in the MME approach is used to choose the weighting matrix and model errors are modeled as time-varying parameters whose variations are chosen to be minimal. The NRMME algorithm coincides with the EKF algorithm when the measurements are linear in the states.

Received 13 June 2002; revision received 29 May 2003; accepted for publication 9 June 2003.

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The simulation results in the paper show that the NRMME is effective and feasible in obtaining optimal estimation for nonlinear systems with nonlinear measurements in the face of significant model errors and large initial condition errors.

The organization of the paper is as follows. First, a detailed derivation of the NRMME algorithm is given. Then, two examples are provided. Finally, discussion of stability and convergence on the NRMME estimator is made and conclusions follow.

Algorithm

Consider a nonlinear discrete-time system whose state dynamics are modeled as

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_0(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{1}$$

where $\boldsymbol{f}_0$ is an $n \times 1$ model vector, $\boldsymbol{x}_k$ is an $n \times 1$ state vector, and $\boldsymbol{u}_k$ is a $q \times 1$ vector denoting uncertain variables in the system model at time k. The components of $\boldsymbol{u}_k$ are modeled as time-varying parameters to be determined in the estimation process. The system of equations for $\boldsymbol{u}_k$ is given by

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + \boldsymbol{d}_k \tag{2}$$

where $\boldsymbol{d}_k$ is a $q \times 1$ vector of bounded model errors with piecewise continuity.

Combining Eqs. (1) and (2) and defining

$$\boldsymbol{X}_k = [\boldsymbol{x}_k \quad \boldsymbol{u}_k]^T$$

yield the augmented equations of the form

$$\boldsymbol{X}_{k+1} = \boldsymbol{f}(\boldsymbol{X}_k) + \boldsymbol{G} \cdot \boldsymbol{d}_k \tag{3}$$

where

$$\boldsymbol{f}(\boldsymbol{X}_k) = [\boldsymbol{f}_0(\boldsymbol{x}_k, \boldsymbol{u}_k)\, \boldsymbol{u}_k]^T, \qquad \boldsymbol{G} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

and where $\boldsymbol{X}_k$ is an $(n+q) \times 1$ augmented state vector, $\boldsymbol{I}$ is a $q \times q$ identity matrix, and $\boldsymbol{0}$ is an $n \times n$ zero matrix. Consider a set of discrete noisy measurements made with the measurement model given by

$$\tilde{\boldsymbol{Z}}_k = \boldsymbol{h}(\boldsymbol{X}_k) + \boldsymbol{v}_k \tag{4}$$

where $\tilde{\boldsymbol{Z}}_k$ is an $m \times 1$ measurement vector, $\boldsymbol{h}(\boldsymbol{X}_k)$ is an $m \times 1$ nonlinear measurement function, and $\boldsymbol{v}_k$ is an $m \times 1$ vector of Gaussian white measurement noise with zero mean and covariance $\boldsymbol{R}$.

Choose a quadratic criterion function

$$J_N = \frac{1}{2}\sum_{k=0}^{N-1}[\tilde{\boldsymbol{Z}}_{k+1} - \boldsymbol{h}(\boldsymbol{X}_{k+1})]^T \boldsymbol{R}^{-1}[\tilde{\boldsymbol{Z}}_{k+1} - \boldsymbol{h}(\boldsymbol{X}_{k+1})] + \frac{1}{2}\sum_{k=0}^{N-1}\boldsymbol{d}_k^T \boldsymbol{W}^{-1}\boldsymbol{d}_k \tag{5}$$

where $\boldsymbol{W}$ is a $q \times q$ positive and nonsingular weighting matrix. As with optimal control theory, the solution of the estimation problem is obtained by minimizing J_N with respect to $\boldsymbol{X}_k$ and $\boldsymbol{d}_k$. According to the discrete Pontryagin's maximum principle, the minimization of J_N leads to the nonlinear TPBVP summarized as

$$\boldsymbol{X}_{k+1} = \boldsymbol{f}(\boldsymbol{X}_k) - \boldsymbol{G}\boldsymbol{W}\boldsymbol{G}^T\boldsymbol{F}^{-T}(\boldsymbol{X}_k)\boldsymbol{\lambda}_k \tag{6a}$$

$$\boldsymbol{\lambda}_{k+1} = \boldsymbol{F}^{-T}(\boldsymbol{X}_k)\boldsymbol{\lambda}_k + \boldsymbol{H}^T(\boldsymbol{X}_{k+1})\boldsymbol{R}^{-1}[\tilde{\boldsymbol{Z}}_{k+1} - \boldsymbol{h}(\boldsymbol{X}_{k+1})] \tag{6b}$$

with boundary conditions

$$\boldsymbol{X}_0 = \text{specified} \tag{6c}$$

$$\boldsymbol{\lambda}_N = 0 \tag{6d}$$

where $\boldsymbol{\lambda}_k$ is an $(n+q) \times 1$ costate vector, $\boldsymbol{H}(\boldsymbol{X}_{k+1}) = \partial\boldsymbol{h}(\boldsymbol{X}_{k+1})/\partial\boldsymbol{X}_{k+1}^T$ is the measurement Jacobian, and $\boldsymbol{F}(\boldsymbol{X}_k) = \partial\boldsymbol{f}(\boldsymbol{X}_k)/\partial\boldsymbol{X}_k^T$ is the dynamics Jacobian.

Notice that, in Eq. (5), the criterion function J_N may be considered as a function of N and $\boldsymbol{X}_N$. Now, N is considered to be the current time, which may assume any value between 0 and ∞, and $\boldsymbol{X}_N$ is the current state of the system. In the following development, $\boldsymbol{\lambda}_N$ will be treated as a variable and N as the running variable. Accordingly, replace the integer variable k in Eqs. (6a) and (6b) with N and rewrite them in a simple form,

$$\boldsymbol{X}_{N+1} = \boldsymbol{\alpha}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) \tag{7}$$

$$\boldsymbol{\lambda}_{N+1} = \boldsymbol{\eta}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) \tag{8}$$

Assume that, for small values of $\boldsymbol{\lambda}_N$,

$$\boldsymbol{X}_N = \hat{\boldsymbol{X}}_N - \boldsymbol{P}_N\boldsymbol{\lambda}_N \tag{9}$$

where $\hat{\boldsymbol{X}}_N$ represents the current state that minimizes the criterion function (5) based on the measurements made over the time interval $[1, N]$ and meets the terminal condition (6d) and where $\boldsymbol{P}_N$ is an $(n+q) \times (n+q)$ matrix. Then, at time $N+1$,

$$\boldsymbol{X}_{N+1} = \hat{\boldsymbol{X}}_{N+1} - \boldsymbol{P}_{N+1}\boldsymbol{\lambda}_{N+1} \tag{10}$$

Substituting Eqs. (7) and (8) into Eq. (10) gives

$$\hat{\boldsymbol{X}}_{N+1} = \boldsymbol{\alpha}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) + \boldsymbol{P}_{N+1}\boldsymbol{\eta}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) \tag{11}$$

Because $\boldsymbol{\lambda}_N$ is small, $\boldsymbol{\alpha}(\boldsymbol{X}_N, \lambda_N)$ and $\boldsymbol{\eta}(\boldsymbol{X}_N, \lambda_N)$ can be approximated in powers of $\boldsymbol{\lambda}_N$, that is,

$$\boldsymbol{\alpha}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) \approx \boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) + \left.\frac{\partial\boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N - \boldsymbol{P}_N\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_N)}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0}\boldsymbol{\lambda}_N \tag{12}$$

$$\boldsymbol{\eta}(\boldsymbol{X}_N, \boldsymbol{\lambda}_N) \approx \boldsymbol{\eta}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) + \left.\frac{\partial\lambda(\hat{\boldsymbol{X}}_N - \boldsymbol{P}_N\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_N)}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0}\boldsymbol{\lambda}_N \tag{13}$$

Substitute Eqs. (12) and (13) into Eq. (11) and note that Eq. (11) must hold for arbitrary $\boldsymbol{\lambda}_N$. Equating the coefficients of the like powers of $\boldsymbol{\lambda}_N$ in Eq. (11) results in a pair of nonlinear difference equations in $\hat{\boldsymbol{X}}_N$ and $\boldsymbol{P}_N$, that is,

$$\hat{\boldsymbol{X}}_{N+1} = \boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) + \boldsymbol{P}_{N+1}\boldsymbol{\eta}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) \tag{14}$$

$$\boldsymbol{P}_{N+1} = -\left.\frac{\partial\boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N - \boldsymbol{P}_N\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_N)}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0} \times \left[\left.\frac{\partial\eta(\hat{\boldsymbol{X}}_N - \boldsymbol{P}_N\boldsymbol{\lambda}_N, \boldsymbol{\lambda}_N)}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0}\right]^{-1} \tag{15}$$

Equations (6a) and (6b) are used to determine $\boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N, \boldsymbol{0})$ and $\boldsymbol{\eta}(\hat{\boldsymbol{X}}_N, \boldsymbol{0})$ in Eq. (14) and the first partial derivatives in Eq. (15). After some algebraic manipulations,

$$\boldsymbol{\alpha}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) = \boldsymbol{f}(\hat{\boldsymbol{X}}_N) \tag{16}$$

$$\boldsymbol{\eta}(\hat{\boldsymbol{X}}_N, \boldsymbol{0}) = \boldsymbol{H}^T[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)]\boldsymbol{R}^{-1}\{\tilde{\boldsymbol{Z}}_{N+1} - \boldsymbol{h}[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)]\} \tag{17}$$

$$\left.\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0} = -\boldsymbol{F}(\hat{\boldsymbol{X}}_N)\boldsymbol{P}_N - \boldsymbol{G}\boldsymbol{W}\boldsymbol{G}^T\boldsymbol{F}^{-T}(\hat{\boldsymbol{X}}_N) \tag{18}$$

$$\left.\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0} = \boldsymbol{F}^{-T}(\hat{\boldsymbol{X}}_N) + \frac{\partial\boldsymbol{M}_{N+1}}{\partial\boldsymbol{f}^T(\hat{\boldsymbol{X}}_N)}\left.\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\lambda}_N^T}\right|_{\lambda_N=0} \tag{19}$$

where

$$\boldsymbol{M}_{N+1} = \boldsymbol{H}^T[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)]R^{-1}\{\tilde{\boldsymbol{Z}}_{N+1} - \boldsymbol{h}[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)]\} \tag{20}$$

$$\boldsymbol{H}[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)] = \frac{\partial\boldsymbol{h}[\boldsymbol{f}(\hat{\boldsymbol{X}}_N)]}{\partial\boldsymbol{f}^T(\hat{\boldsymbol{X}}_N)} \tag{21}$$

Define

$$\hat{X}_{N+1|N} \equiv f(\hat{X}_N) \tag{22a}$$

From Eq. (14), the measurement update for the state estimate is

$$\hat{X}_{N+1} = \hat{X}_{N+1|N} + P_{N+1}H^T[f(\hat{X}_N)]R^{-1}\{\tilde{Z}_{N+1} - h[f(\hat{X}_N)]\} \tag{22b}$$

Defining

$$P_{N+1|N} \equiv F(\hat{X}_N)P_N F^T(\hat{X}_N) + GWG^T \tag{23a}$$

and substituting Eqs. (18) and (19) into Eq. (15) yield

$$P_{N+1} = P_{N+1|N}\left(I - \frac{\partial M_{N+1}}{\partial \hat{X}^T_{N+1|N}} P_{N+1|N}\right)^{-1} \tag{23b}$$

The weighting matrix W in Eq. (23a) is assumed to be diagonal and to satisfy the covariance constraint condition conceptualized by Mook and Junkins[9] and to be given by

$$\frac{1}{N}\sum_{k=1}^{N}(\tilde{Z}_k - \hat{Z}_k)(\tilde{Z}_k - \hat{Z}_k)^T \approx R \tag{24}$$

Define

$$S_k = \frac{1}{k}\sum_{i=1}^{k}\varepsilon_i\varepsilon_i^T \tag{25}$$

where $\{\varepsilon_i = \tilde{Z}_i - \hat{Z}_i\}$ is an $m \times 1$ vector sequence of the measurement residuals. The sum matrix S_k can be computed recursively according to

$$S_{k+1} = [k/(k+1)]\,S_k + [1/(k+1)]\varepsilon_{k+1}\varepsilon_{k+1}^T \tag{26}$$

The covariance constraint implies that, when the estimate has converged, the residual sequence $\{\varepsilon_i\}$ will tend to be a zero-mean, white Gaussian noise with covariance R. There is no efficient way of choosing an optimal weighting matrix W that satisfies the covariance constraint, and so heuristic methods are often used. However, experience has shown that the stability and convergence of the NRMME algorithm are not strongly sensitive to minor variations of W.

Substituting Eq. (20) into Eq. (23b) and expanding it give

$$P_{N+1} = P_{N+1|N}\big[I + H^T(\hat{X}_{N+1|N})R^{-1}H(\hat{X}_{N+1|N})P_{N+1|N} - A_{N+1|N}\big]^{-1} \tag{27}$$

with

$$A_{N+1|N} = \left(\sum_{j=1}^{m}\frac{\partial a_j}{\partial \hat{X}^T_{N+1|N}}\delta_j\right)P_{N+1|N} \tag{28}$$

where a_j is the jth column of $H(\hat{X}_{N+1|N})$ and δ_i is the jth scalar component of the output-prediction error vector $\delta = \tilde{Z}_{N+1} - h[f(\hat{X}_N)]$. Note that inclusion of the higher-order term (HOT) $A_{N+1|N}$ in Eq. (27) may cause P_N to be negative. It is intuitive that $A_{N+1|N}$ can be dropped as done in Ref. 14 to keep P_N nonnegative. Then, the NRMME algorithm reduces to the EKF algorithm. However, the simulation study shows that, when P_N is positive, inclusion of $A_{N+1|N}$ in Eq. (27) may contribute to the unbiased estimation when initial condition errors are large.

It is apparent that, from derivation of the algorithm, the estimate for the NRMME approach approximates the optimal solution of the nonlinear TPBVP that minimizes the criterion. The NRMME algorithm can be implemented sequentially. A higher-order approximation can be made with much complexity and high computational burden.

Simulation Results

Scalar Example

To illustrate the application of the NRMME approach, consider the estimation of the state history of a scalar example in Ref. 15. The discrete-time equation of motion is

$$x_{k+1} = x_k - T\sin x_k - \tfrac{1}{2}T^2\cos x_k + d_k \tag{29}$$

where the model error d_k is assumed as a zero-mean, Gaussian process with known variance of q^2. A nonlinear measurement equation is of the form

$$z_k = \tfrac{1}{2}\sin(2x_k) + \nu_k, \qquad \nu_k \in N(0, r) \tag{30}$$

which is highly nonlinear and periodic in x_k. The weighting matrix in the NRMME algorithm is chosen as q^2 so that the NRMME is a maximum likelihood estimation. To examine the influence of the HOT in Eq. (27) on convergence of the NRMME method, 100 Monte Carlo runs were performed with identical noise sequence and with the same initial conditions for both cases with the HOT and without the HOT. Notice that without the HOT, the NRMME algorithm is reduced to the EKF algorithm. The initial conditions of 5.4 and 4.6 rad were used against the true initial values of 4.4 rad. Figure 1 shows comparison of the state estimates for the initial conditions of 5.4 rad. The comparison implies that inclusion of the HOT helps the estimator converge for large initial condition errors. Figure 2 shows that the estimates converge for small initial condition errors for both with the HOT and without the HOT. The simulation results for this example are consistent with the well-accepted conclusion on the convergence of the EKF for different initial conditions.

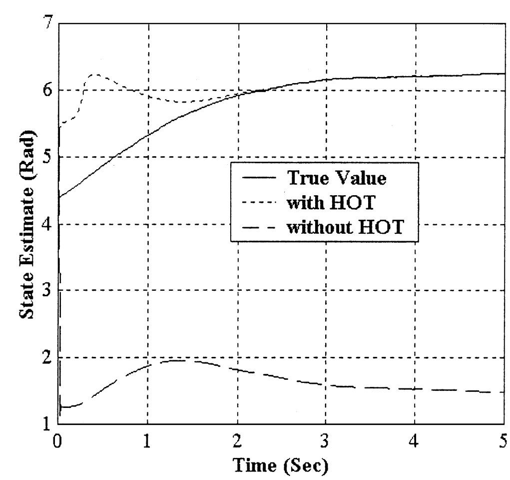

Fig. 1 State estimates averaged over 100 runs for initial conditions of 5.4 rad.

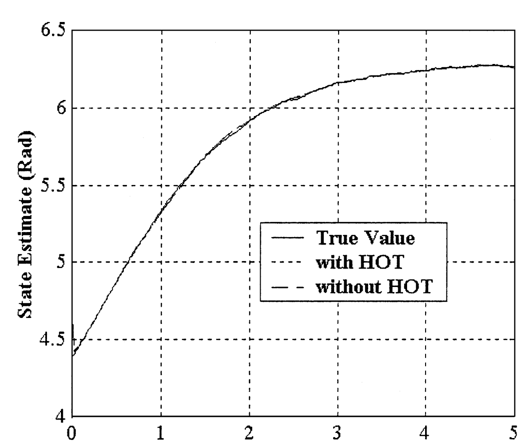

Fig. 2 State estimates averaged over 100 runs for initial conditions of 4.6 rad.

Application to Compatibility Checking of Flight-Test Data

The angular rate and the attitude angle are sometimes measured simultaneously in flight tests, but the measured quantities may deviate from the real value due to instrument errors. The instrument errors can be removed to reconstruct the real attitude and angular-rate value by using the redundant information available in the measurements. The data are reconstructed by formulating an exact system equation relating the attitude quaternion and the angular rate and by including the angular-rate measurement error as an uncertain variable to be determined and the attitude measurement bias as an unknown constant parameter. Reconstruction of the attitude angle and the angular rate is essentially a joint parameter and state estimation problem with a nonlinear system expressed as a set of equations of the form

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} \tilde{\omega}_{xb} - \Delta\omega_{xb} \\ \tilde{\omega}_{yb} - \Delta\omega_{yb} \\ \tilde{\omega}_{zb} - \Delta\omega_{zb} \end{bmatrix} \tag{31}$$

$$\dot{\boldsymbol{b}} = \boldsymbol{0} \tag{32}$$

$$\Delta\dot{\boldsymbol{\omega}} = \boldsymbol{d} \tag{33}$$

where q_0, q_1, q_2, and q_3 are quaternion components; $\tilde{\omega}_{xb}$, $\tilde{\omega}_{yb}$, and $\tilde{\omega}_{zb}$ are the measured angular-rate components; $\boldsymbol{b} = (b_\vartheta, b_\psi, b_\gamma)^T$ is the bias vector of the attitude measurement; and $\Delta\boldsymbol{\omega} = (\Delta\omega_{xb}, \Delta\omega_{yb}, \Delta\omega_{zb})^T$ is the angular-rate measurement error vector, which is assumed to be dynamic. The model error vector $\boldsymbol{d}$ in Eq. (33) represents the uncertainty of $\Delta\boldsymbol{\omega}$. A set of state-observable measurements were modeled by the nonlinear system of equations for the pitch angle ϑ, the yaw angle ψ, and the roll angle γ,

$$\vartheta = \sin^{-1}[2(q_1q_2 + q_0q_3)] + b_\vartheta + \nu_1 \tag{34}$$

$$\psi = \tan^{-1}\frac{2(q_0q_2 - q_1q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2} + b_\psi + \nu_2 \tag{35}$$

$$\gamma = \tan^{-1}\frac{2(q_0q_1 - q_2q_3)}{q_0^2 - q_1^2 + q_2^2 - q_3^2} + b_\gamma + \nu_2 \tag{36}$$

To maintain quaternion normalization, the normality constraint was introduced as a pseudomeasurement,

$$z_q = q_0^2 + q_1^2 + q_2^2 + q_3^2 + \nu_4 \tag{37}$$

where z_q, simulated by $1 + \nu_4$, is the pseudomeasurement of the squared sum of the quaternion and ν_4 is the associated deviation of the estimated quaternion from the normality constraint. In Eqs. (34–37), ν_1, ν_2, ν_3, and ν_4 are assumed as components of a zero-mean, Gaussian white measurement noise vector with covariance matrix $\boldsymbol{R}$. From Eqs. (34–37), compatibility checking of flight-test data is a highly nonlinear estimation problem.

The NRMME algorithm was used to evaluate this estimation problem. A six-degrees-of-freedom movement of the aircraft dynamics was simulated to provide the needed data for simulation studies. The components of $\Delta\boldsymbol{\omega}$ were simulated as the dynamic errors caused by scale factors and were intentionally chosen to be much larger than the nominal value for ordinary rate gyros. The bias components were all set to 5.7 deg. The initial estimates of $\Delta\boldsymbol{\omega}$ and the bias were all set to zero.

The simulated and estimated trajectories of the attitude and the estimation error plotted in Fig. 3 show the attitude estimates by the NRMME. The estimated and simulated angular-rate measurement errors are plotted in Fig. 4. Random noise was included in the simulated measurement errors for the angular rate as shown in Fig. 4. The estimates of $\Delta\boldsymbol{\omega}$ have much smaller variations than the simulated results. The estimated bias is plotted in Fig. 5. The rapid convergence of the estimated component b_γ from the initial error is also shown in Fig. 5.

Discussion of Stability and Convergence

To begin, the stability and convergence of the estimation equations (22a) and (22b) require that the estimated system itself be at least asymptotically stable and observable.

Let

$$\tilde{\boldsymbol{e}}_N = \boldsymbol{X}t_N - \hat{\boldsymbol{X}}_N \tag{38}$$

where $\boldsymbol{X}t_N$ is the true value of the state and $\tilde{\boldsymbol{e}}_N$ is the state estimate error at time N. For small values of $\tilde{\boldsymbol{e}}_N$, an approximate evolution equation for $\tilde{\boldsymbol{e}}_N$ can be derived as

$$\tilde{\boldsymbol{e}}_{N+1} \approx \left[\boldsymbol{I} - \boldsymbol{P}_{N+1}\boldsymbol{H}_{N+1}^T\boldsymbol{R}^{-1}\boldsymbol{H}_{N+1}\right]\boldsymbol{F}_N\tilde{\boldsymbol{e}}_N - \boldsymbol{P}_{N+1}\boldsymbol{H}_{N+1}^T\boldsymbol{R}^{-1}\boldsymbol{\nu}_{N+1} + \boldsymbol{G}_N\boldsymbol{d}_N \tag{39}$$

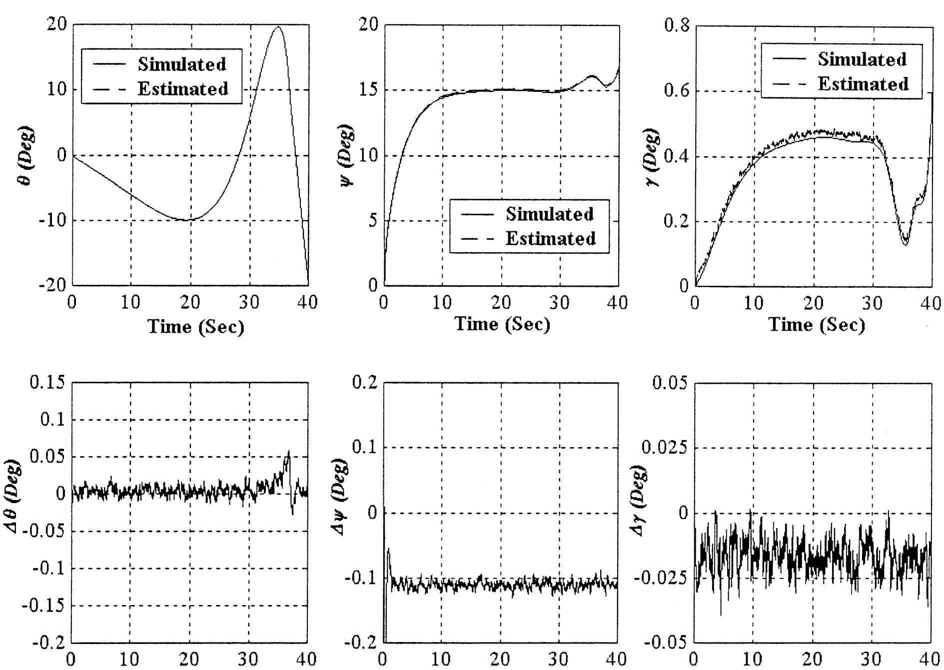

Fig. 3 Attitude estimates and estimation errors for flight-test data with instrument errors.

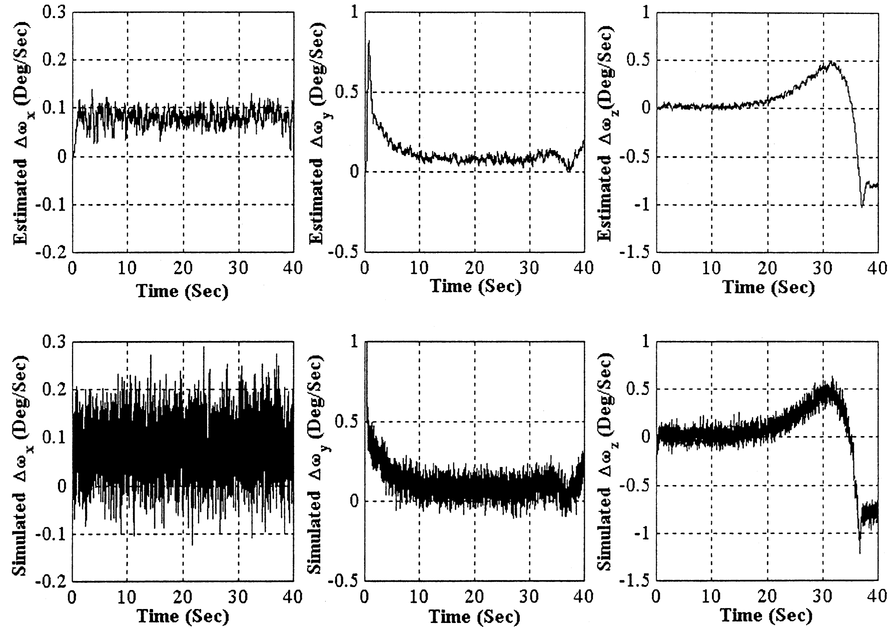

Fig. 4 Estimated and simulated angular-rate measurement errors for the assumed rate gyro.

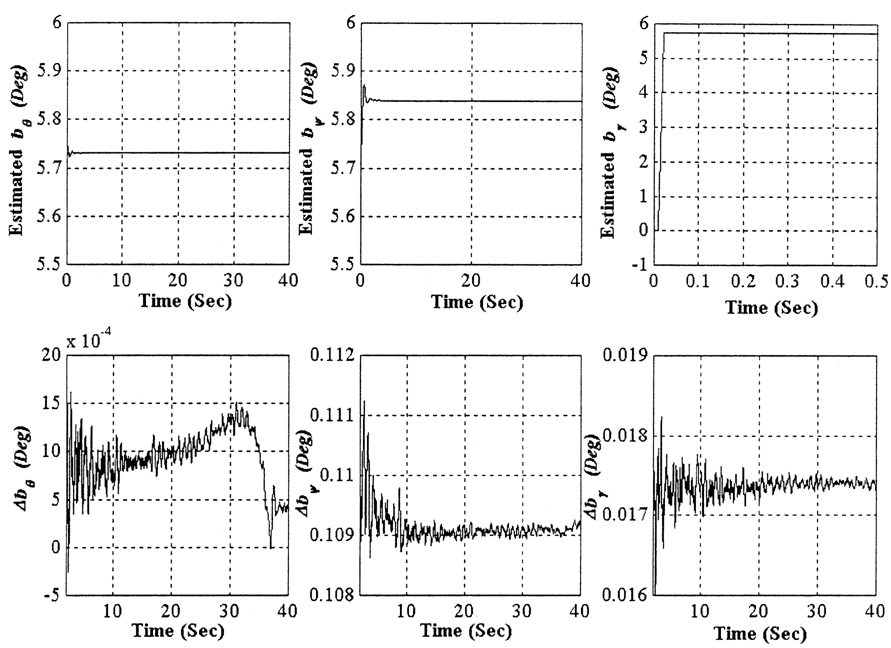

Fig. 5 Estimated angular measurement bias and estimation error for the assumed attitude sensor.

Equation (39) is a linear time-varying stochastic difference equation in $\tilde{e}_N$. If the homogeneous part of Eq. (39) is stable, the estimate error $\tilde{e}_N$ will be bounded. Because $H_{N+1}^T R^{-1} H_{N+1} > 0$, Eq. (39) will be unstable if $P_{N+1} < 0$. Besides, there is an upper bound on P_{N+1} to stabilize Eq. (39). In a simple case, for example, where F_N is an identity matrix, it is required that

$$P_{N+1} \leq \left(H_{N+1}^T R^{-1} H_{N+1}\right)^{-1} \tag{40}$$

Designate the error covariance associated with $\hat{X}_N$ as S_N. Given that the covariance constraint condition (24) is satisfied, the evolution equation of S_N will take the form

$$S_{N+1} \approx \left[I - P_{N+1} H_{N+1}^T R^{-1} H_{N+1}\right] F_N S_N F_N^T \times \left[I - P_{N+1} H_{N+1}^T R^{-1} H_{N+1}\right]^T + P_{N+1} H_{N+1}^T H_{N+1} P_{N+1}^T + G_N Q_N G_N^T \tag{41}$$

where $Q_N = E\{d_N d_N^T\}$. Because the driving terms in Eq. (41) are always nonnegative, it is expected that if the associated Eq. (39) is stable, the error covariance will be bounded.

Because of the driving terms containing the factor $\tilde{Z}_{N+1} - h(\hat{X}_{N+1|N})$ in the state estimation equations and the sequential nature of the estimation equations, the NRMME estimator will provide a set of the improved state estimates that fit the real value optimally in statistical sense if the estimation equations converge. Although the NRMME algorithm represents an approximate solution to the minimization problem, rigorous analysis of its stability and convergence with unknown or uncertain initial conditions and noisy measurements will be much difficult. The discussion in this section is only a qualitative analysis under the assumption that the estimate errors are small. Theoretical difficulites with stability analysis of nonlinear stochastic difference equations and convergence of the stochastic sequence are major obstacles, especially when the statistical or stochastic characteristics of the model errors are unknown. A simulation study seems to be a practical way to assess the feasibility and effectiveness of the estimator for specific applications.

Conclusions

The NRMME estimator is a new technique for recursive estimation problems with model uncertainties and nonlinear measurements. The technique of invariant imbedding was used to find the solution to the nonlinear TPBVP resulting from the minimization of the criterion function with respect to the model error. The resulting algorithm can be implemented in real time. The covariance constraint condition put forward in the MME estimation was employed to choose the weighting matrix for the model error. When measurements are linear in the states or the term containing the second partials of the output function is dropped, the proposed algorithm coincides with that of the EKF for state and parameter estimation. Performance was verified by two examples. Results show that the state estimates given by the proposed method are stable and convergent in the presence of significant model errors and nonlinear measurements. A qualitative discussion on stability and convergence was made for small estimate errors.

The method is potentially applicable to areas such as attitude determination and orbit estimation in the aerospace industry, real-time tracking and prediction of dynamic systems, and online industrial control, where nonlinearities and model uncertainties prevail in actual practice. However, further work is needed to analyze asymptotic properties, to choose efficiently the weighting matrix $\boldsymbol{W}$, and to estimate with non-Gaussian measurement noise.

References

[1]Kasdin, N. J., and Haupt, G. T., "Second-Order Correction and Numerical Considerations for the Two-Step Optimal Estimator," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 2, 1997, pp. 362–369.

[2]Lisano, M. E., "Comment on 'New, Guaranteed Positive Time Update for the Two-Step Optimal Estimator,'" *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 6, 2000, pp. 1088–1090.

[3]Kenney, J. D., Beard, R., and Stirling, W., "Set-Valued Nonlinear Estimation Using the Galerkin Approximation," *Proceedings of the American Control Conference*, American Automatic Control Conference, Evanston, IL, 1998, pp. 3580–3584.

[4]Wan, E. A., and Merwe, R., "The Unscented Kalman Filter for Nonlinear Estimation," *Adaptive Systems for Signal Processing, Communications and Control Symposium 2000*, IEEE Press, Piscataway, NJ, 2000, pp. 153–158.

[5]Detchmendy, D. M., and Sridhar, R., "Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems," *Proceedings 1965 Joint Automatic Control Conference*, American Automatic Control Conference, Evanston, IL, 1965, pp. 56–63.

[6]Bellman, R. E., Kagiwada, H. H., Kalaba, R. E., and Sridhar, R., "Invariant Imbeedding and Nonlinear Filtering Theory," *Journal of the Astronautical Sciences*, Vol. 13, No. 3, 1966, pp. 110–115.

[7]Pearson, J. B., "On Nonlinear Least-Squares Filtering," *Automatica*, Vol. 4, 1967, pp. 97–105.

[8]Sage, A. P., *Optimum Systems Control*, Prentice–Hall, Englewood Cliffs, NJ, 1968, Chap. 15.

[9]Mook, D. J., and Junkins, J. L., "Minimum Model Error Estimation for Poorly Modeled Dynamic Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 3, No. 4, 1988, pp. 367–375.

[10]Mook, D. J., "Existence and Uniqueness Proof for the Minimum Model Error Optimal Estimation Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 1064–1067.

[11]Crassidis, J. L., and Markley, F. L., "Minimum Model Error Approach for Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 6, 1997, pp. 1241–1247.

[12]Crassidis, J. L., and Markley, F. L., "Predictive Filtering for Nonlinear Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, 1997, pp. 566–572.

[13]Lu, P., "Nonlinear Predictive Controllers for Continuous Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 3, 1994, pp. 553–560.

[14]Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970, Chap. 9.

[15]Gelb, A., *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974, Chap. 6.